

On irreducible algebraic sets over linearly ordered semilattices

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Abstract

Equations over linearly ordered semilattices are studied. For any equation $t(X) = s(X)$ we find irreducible components of its solution set and compute the average number of irreducible components of all equations in n variables.

1 Introduction

This paper is devoted to the following problem. One can define a notion of an equation over a linearly ordered semilattice $L_l = \{a_1, a_2, \dots, a_l\}$ (the formal definition of an equation is given below). A set Y is *algebraic* if it is the solution set of some system of equations over L_l . Let us consider an equation $t(X) = s(X)$ over L_l , and Y be the solution set of $t(X) = s(X)$. One can find algebraic sets Y_1, Y_2, \dots, Y_m such that $Y = \bigcup_{i=1}^m Y_i$. One can decompose each Y_i into a union of other algebraic sets, etc. This process terminates after a finite number of steps and gives a decomposition of Y into a union of *irreducible* algebraic sets Y_i (the sets Y_i are called the *irreducible components* of Y). Roughly speaking, irreducible algebraic sets are “atoms” which form any algebraic set. The size and the number of such “atoms” are important characteristics of the semilattices L_l , since there are connections between irreducible algebraic sets and universal theory of linearly ordered semilattices (see [1]). Moreover, the number of irreducible components was involved in the estimation of lower bounds of algorithm complexity (see [2] for more details).

In this paper (Section 4) we study the properties of the irreducible components of the solution set Y of an equation $t(X) = s(X)$. Precisely, we prove that the union of irreducible algebraic sets $Y = \bigcup_{i=1}^m Y_i$ is redundant, i.e. the intersections $\bigcap_{i \in I} Y_i$ ($|I| < m$) consists of many points (Proposition 4.5). Moreover, for any equation $t(X) = s(X)$ in n variables we count the number m of irreducible components (see (6)), and in Section 5 we count the average number $\overline{\text{Irr}}(n, l)$ of irreducible components of the solution sets of equations in n variables.

2 Main definitions

Let $L_l = \{a_1, a_2, \dots, a_l\}$ be the linearly ordered semilattice of l elements and $a_1 < a_2 < \dots < a_l$. The multiplication in L_l is defined by $a_i \cdot a_j = a_{\min(i,j)}$. Obviously, the linear order on L_l can be expressed by the multiplication as follows

$$a_i \leq a_j \Leftrightarrow a_i a_j = a_i.$$

A *term* $t(X)$ in variables $X = \{x_1, x_2, \dots, x_n\}$ is a commutative word in letters x_i .

Let $\text{Var}(t)$ be the set of all variables occurring in a term $t(X)$. Following [1], an *equation* is an equality of terms $t(X) = s(X)$. Below we consider inequalities $t(X) \leq s(X)$ as equations, since $t(X) \leq s(X)$ is the short form of $t(X)s(X) = t(X)$. Notice that we consider equations as *ordered pairs* of terms, i.e. the expressions $t(X) = s(X)$, $s(X) = t(X)$ are *different* equations. Let $Eq(n)$ denote the set of all equations in $X = \{x_1, x_2, \dots, x_n\}$ variables (we assume that each $t(X) = s(X) \in Eq(n)$ contains the occurrences of all variables x_1, x_2, \dots, x_n). An equation $t(X) = s(X) \in Eq(n)$ is said to be a (k_1, k_2) -*equation* if $|\text{Var}(t) \setminus \text{Var}(s)| = k_1$ and $|\text{Var}(s) \setminus \text{Var}(t)| = k_2$. For example, $x_1x_2 = x_1x_3x_4$ is a $(1, 2)$ -equation. Let $Eq(k_1, k_2, n) \subseteq Eq(n)$ be the set of all (k_1, k_2) -equations in n variables. Obviously,

$$Eq(n) = \bigcup_{(k_1, k_2) \in K_n} Eq(k_1, k_2, n), \quad (1)$$

where

$$K_n = \{(k_1, k_2) \mid k_1 + k_2 \leq n\} \setminus \{(0, n), (n, 0)\}.$$

Each equation $t(X) = s(X) \in Eq(k_1, k_2, n)$ is uniquely defined by k_1 variables in the left part and by k_2 other variables in the right part (the residuary $n - k_1 - k_2$ variables should occur in both parts of the equation). Thus,

$$\#Eq(k_1, k_2, n) = \binom{n}{k_1} \binom{n - k_1}{k_2}.$$

By (1), one can compute

$$\#Eq(n) = 3^n - 2.$$

Remark 2.1. In this paper we consider only equations $t(X) = s(X)$ with $n > l$, i.e. the number of variables occurring in $t(X) = s(X)$ is more than the order of the semilattice L_l . The case $n \leq l$ needs the different technic and was announced in [3].

A point $P \in L_l^n$ is a *solution* of an equation $t(X) = s(X)$ if $t(P), s(P)$ define the same element in the semilattice L_l . By the properties of linearly ordered semilattices, a point $P = (p_1, p_2, \dots, p_n)$ is a solution of $t(X) = s(X)$ iff there exist variables $x_i \in \text{Var}(t)$, $x_j \in \text{Var}(s)$ such that $p_i = p_j$ and $p_i \leq p_k$ for all $1 \leq k \leq n$. The set of all solutions of an equation $t(X) = s(X)$ is denoted by $V(t(X) = s(X))$.

An arbitrary set of equations is called a *system*. The set of all solutions $V(\mathbf{S})$ of a system $\mathbf{S} = \{t_i(X) = s_i(X) \mid i \in I\}$ is defined as $\bigcap_{i \in I} V(t_i(X) = s_i(X))$. A set $Y \subseteq L_l^n$ is called *algebraic over L_l* if there exists a system \mathbf{S} in n variables with $V(\mathbf{S}) = Y$. An algebraic set Y is *irreducible* if Y is not a proper finite union of other algebraic sets.

Proposition 2.2. *Any algebraic set Y over L_l is a finite union of irreducible sets*

$$Y = Y_1 \cup Y_2 \cup \dots \cup Y_m, \quad Y_i \not\subseteq Y_j \text{ for all } i \neq j, \quad (2)$$

and this decomposition is unique up to a permutation of components.

Proof. A semilattice S is *equationally Noetherian* if for any infinite system \mathbf{S} in variables $X = \{x_1, x_2, \dots, x_n\}$ there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with the same solution set. According to [1], the decomposition (2) holds for any algebraic set Y over an equationally Noetherian semilattice S . Thus, it is sufficient to prove that L_l is equationally Noetherian.

However the condition $|Eq(n)| < \infty$ gives that there is not any infinite system over L_l . Thus, L_l is equationally Noetherian. \square

The subsets Y_i from the union (2) are called the *irreducible components* of Y .

Let Y be an algebraic set over L_l defined by a system $\mathbf{S}(X)$. One can define an equivalence relation \sim_Y over the set of all terms in variables X as follows

$$t(X) \sim_Y s(X) \Leftrightarrow t(P) = s(P) \text{ for any point } P \in Y.$$

The set of \sim_Y -equivalence classes is called *the coordinate semilattice of Y* and denoted by $\Gamma(Y)$ (see [1] for more details). The following statement describes the coordinate semilattices of irreducible algebraic sets.

Proposition 2.3. *A set Y is irreducible over L_l iff $\Gamma(Y)$ is embedded into L_l*

Proof. Following [1], $\Gamma(Y)$ is discriminated by L_l iff Y is irreducible (see [1] for the definition of the discrimination). However for a finite semilattice L_l the discrimination is equivalent to the embedding. \square

There are different algebraic sets over L_l with isomorphic coordinate semilattices. Such sets are called *isomorphic*. For example, the following sets

$$Y_1 = V(\{x_1 \leq x_2 \leq x_3\}), \quad Y_2 = V(\{x_3 \leq x_2 \leq x_1\})$$

has the isomorphic coordinate semilattices

$$\Gamma(Y_1) = \langle x_1, x_2, x_3 \mid x_1 \leq x_2 \leq x_3 \rangle \cong L_3,$$

$$\Gamma(Y_2) = \langle x_1, x_2, x_3 \mid x_3 \leq x_2 \leq x_1 \rangle \cong L_3.$$

Thus, Y_1, Y_2 are isomorphic.

3 Example

Let $n = 3, l = 2$. We have exactly $Eq(3) = 3^3 - 2 = 25$ equations in three variables over L_2 . The following table contains the information about such equations over L_2 . The second column contains systems which define irreducible components of the solution set of an equation in the first column. A cell of the table contains \uparrow if an information in this cell is similar to the cell above.

Equations	Irreducible components (IC)	Number of IC
$x_1x_2x_3 = x_1x_2x_3$	$x_1 \leq x_2 = x_3 \cup x_1 = x_2 \leq x_3 \cup$ $x_2 \leq x_1 = x_3 \cup x_3 \leq x_1 = x_2 \cup$ $x_1 = x_3 \leq x_2 \cup x_2 = x_3 \leq x_1$	6
$x_1 = x_1x_2x_3,$ $x_1x_2x_3 = x_1$	$x_1 \leq x_2 = x_3 \cup x_1 = x_2 \leq x_3 \cup$ $x_1 = x_3 \leq x_2$	3
$x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_2$	\uparrow	3
$x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_3$	\uparrow	3
$x_1 = x_2x_3,$ $x_2x_3 = x_1$	$x_1 = x_2 \leq x_3 \cup x_1 = x_3 \leq x_2$	2
$x_2 = x_1x_3,$ $x_1x_3 = x_2$	\uparrow	2
$x_3 = x_1x_2,$ $x_1x_2 = x_3$	\uparrow	2
$x_1x_2 = x_1x_3,$ $x_1x_3 = x_1x_2$	$x_1 = x_2 \leq x_3 \cup x_1 = x_3 \leq x_2 \cup$ $x_1 \leq x_2 = x_3 \cup x_2 = x_3 \leq x_1$	4
$x_1x_2 = x_2x_3,$ $x_2x_3 = x_1x_2$	\uparrow	4
$x_1x_3 = x_2x_3,$ $x_2x_3 = x_1x_3$	\uparrow	4
$x_1x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_2$	$x_1 = x_2 \leq x_3 \cup x_1 = x_3 \leq x_2 \cup$ $x_1 \leq x_2 = x_3 \cup x_2 = x_3 \leq x_1 \cup$ $x_2 \leq x_1 = x_3$	5
$x_1x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_3$	\uparrow	5
$x_2x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_2x_3$	\uparrow	5

One can directly compute the average number of irreducible components of algebraic sets defined by equations in three variables:

$$\overline{\text{Irr}}(3, 2) = \frac{6 + 2(3 + 3 + 3 + 2 + 2 + 2 + 4 + 4 + 4 + 5 + 5 + 5)}{25} = \frac{90}{25} = 3.6 \quad (3)$$

Recall that in Section 5 we obtain the general expression for $\overline{\text{Irr}}(n, l)$ (7). Clearly, (7) gives (3) for $n = 3, l = 2$ (see the proof in (8) and (9)).

4 Decompositions of algebraic sets

Let Y denote the solution set of an equation $t(X) = s(X)$ over the semilattice $L_l = \{a_1, a_2, \dots, a_l\}$. The table above shows that any irreducible component divides the variables X into l classes and sorts the classes in some order. The following definition formalizes such properties of irreducible components.

A disjoint partition $\sigma = (X_1, X_2, \dots, X_l)$ of the set $X = \{x_1, x_2, \dots, x_n\}$ is called *ordered* if there is a linear order \leq_σ on σ : $X_1 \leq_\sigma X_2 \leq_\sigma \dots \leq_\sigma X_l$. Let $\chi_\sigma(x_i)$ denote the class X_k with $x_i \in X_k$.

We shall denote $x_i =_\sigma x_j$ ($x_i \leq_\sigma x_j$) if $\chi(x_i) = \chi(x_j)$ (respectively, $\chi_\sigma(x_i) \leq_\sigma \chi_\sigma(x_j)$).

An ordered partition σ is *Y-irreducible* if the set X_1 (the minimal set of the order \leq_σ) contains a variable from $t(X)$ and a variable from $s(X)$.

For example, an equation $x_1x_2x_3 = x_1$ over L_2 has the following Y-irreducible partitions: $(\{x_1\}, \{x_2, x_3\})$, $(\{x_1, x_2\}, \{x_3\})$, $(\{x_1, x_3\}, \{x_2\})$. Such partitions obviously correspond to irreducible components of $V(x_1x_2x_3 = x_1)$ in the table above.

Any Y-irreducible partition σ defines an algebraic set Y_σ as follows

$$Y_\sigma = V(\mathbf{S}_\sigma) = V\left(\bigcup_{x_i =_\sigma x_j} \{x_i = x_j\} \bigcup_{x_i <_\sigma x_j} \{x_i \leq x_j\}\right).$$

For example, the partition $\sigma = (\{x_2, x_3\}, \{x_1\})$ defines the system

$$\mathbf{S}_\sigma = \{x_2 = x_3, x_2 \leq x_1, x_3 \leq x_1\}.$$

for $Y = V(\{x_1x_2 = x_1x_3\})$.

Lemma 4.1. *The set Y_σ defined by a Y-irreducible partition σ is an irreducible algebraic set, and moreover $\Gamma(Y_\sigma) \cong L_l$.*

Proof. By the definition of a coordinate semilattice, $\Gamma(Y_\sigma)$ is generated by the elements $\{x_1, x_2, \dots, x_n\}$ and has the following defined relations

$$\{x_i = x_j \mid \text{if } x_i =_\sigma x_j\} \cup \{x_i \leq x_j \mid \text{if } x_i \leq_\sigma x_j\}.$$

It is easy to see that all elements x_i are linearly ordered in $\Gamma(Y_\sigma)$. Thus, $\Gamma(Y_\sigma)$ is a linearly ordered semilattice, and it is isomorphic to L_l . By Proposition 2.3, the set Y_σ is irreducible. \square

The following lemma gives the decomposition of the set $Y = V(t(X) = s(X))$ via ordered partitions.

Lemma 4.2. *The set $Y = V(t(X) = s(X))$ is a union*

$$Y = \bigcup_{\sigma \text{ is } Y\text{-irreducible}} Y_\sigma \quad (4)$$

Proof. Let $P = (p_1, p_2, \dots, p_n) \in Y$. One can define an equivalence relation \sim_P as follows

$$x_i \sim_P x_j \Leftrightarrow p_i = p_j.$$

Thus, we obtain equivalence classes $\{X_1^P, X_2^P, \dots, X_k^P\}$. Since $p_i \in L_l$, $k \leq l$. One can define a linear order $x_i \leq_P x_j$ if $p_i \leq p_j$. The order \leq_P induces a linear order over the classes $\{X_i\}$. Let us fix a pair of variables $x_t, x_s \in X_1^P$ (probably, x_t, x_s is the same variable) such that $x_t \in \text{Var}(t)$ and $x_s \in \text{Var}(s)$ (such pair (x_t, x_s) always exists, since P satisfies the equation $t(X) = s(X)$). Let us find a set Y_σ with $P \in Y_\sigma$ by the following procedure.

Procedure

Input: a set of k equivalence classes $\sigma_0 = (X_1^P, X_2^P, \dots, X_k^P)$ with the linear order \leq_P .

Output: $\sigma = (X_1, X_2, \dots, X_l)$ with a linear order \leq_σ .

Step 0: Put $\sigma = \sigma_0$. If $l = k$ terminate the procedure, otherwise go to the step 1.

Step j ($1 \leq j \leq l - k$):

1. Take an arbitrary equivalence class $X_i \in \sigma = (X_1, X_2, \dots, X_{k+j-1})$ such that $|X_i| \geq 2$ and X_i contains a variable $x \in X \setminus \{x_t, x_s\}$. Such class always exists, since $n > l > k + j - 1$.
2. Move x from X_i to a new class X' and define a linear order \leq_σ by $X_i \leq_\sigma X' \leq X_{i+1}$. Put $\sigma = (X_1, X_2, \dots, X_i, X', X_{i+1}, \dots, X_{l+j-1})$. Go to the next step.

Roughly speaking, the procedure increases the number of classes preserving the relation $<_\sigma$.

After the procedure we obtain an ordered partition σ of l equivalence classes X_i . The procedure does not move the variables x_t, x_s , therefore $x_t, x_s \in X_1$ and σ is a Y -irreducible partition.

Let us prove $P \in Y_\sigma = V(\mathbf{S}_\sigma)$. An equation $x_i \leq x_j \in \mathbf{S}_\sigma$ (one can similarly consider an equality $x_i = x_j \in \mathbf{S}_\sigma$) is not satisfied by P if $p_i > p_j$ or equivalently $x_j <_P x_i$. Since the procedure preserves the relation $<_\sigma$, we have $x_j <_\sigma x_i$, and by the definition of \mathbf{S}_σ , the equation $x_i \leq x_j$ can not occur in \mathbf{S}_σ . Thus, we came to the contradiction.

Let us prove now $Y_\sigma \subseteq Y$ for each σ . Consider a point $P = (p_1, p_2, \dots, p_n) \in Y_\sigma$. Since $\sigma = (X_1, X_2, \dots, X_l)$ is a Y -irreducible partition, the class X_1 contains variables $x_t \in \text{Var}(t)$, $x_s \in \text{Var}(s)$ and $p_t = p_s$. Since X_1 is the minimal class of the order \leq_σ ,

$$x_t \leq x_i \in \mathbf{S}_\sigma, \quad x_s \leq x_i \in \mathbf{S}_\sigma \text{ for any } i \in [1, n] \setminus \{t, s\}.$$

Thus, $p_t = p_s \leq p_i$ for any $1 \leq i \leq n$, and we have

$$t(P) = p_t = p_s = s(P) \Rightarrow P \in V(t(X) = s(X)) = Y.$$

□

Let $\sigma = (X_1, X_2, \dots, X_l)$ be a Y -irreducible partition of X . Let us define a point $P_\sigma = (p_1, p_2, \dots, p_n) \in L_l^n$ by

$$p_i = a_k \text{ if } x_i \in X_k.$$

Lemma 4.3. *The point P_σ belongs to the set Y_σ , and $P_\sigma \notin Y_{\sigma'}$ for each Y -irreducible partition $\sigma' \neq \sigma$. Thus, in the union (4) $Y_\sigma \not\subseteq Y_{\sigma'}$ for distinct partitions σ, σ' .*

Proof. One can directly prove that $P_\sigma \in V(\mathbf{S}_\sigma) = Y_\sigma$.

Let us take an irreducible partition

$$\sigma' = (X'_1, X'_2, \dots, X'_l) \neq \sigma = (X_1, X_2, \dots, X_l).$$

There exist variables x_i, x_j such that $x_i <_\sigma x_j$ but $x_i \geq_{\sigma'} x_j$. For the point P_σ we have $p_i < p_j$, therefore P_σ does not satisfy the equation $x_i \geq x_j \in \mathbf{S}_{\sigma'}$, and $P_\sigma \notin Y_{\sigma'}$. □

According to Lemmas 4.1, 4.2, 4.3, we obtain the following statement.

Theorem 4.4. *The number of Y -irreducible partitions of a set $Y = V(t(X) = s(X))$ is equal to the number of irreducible components of Y .*

The next statement describes the properties the union (4).

Proposition 4.5. *Let (4) be a union of the irreducible components of a set $Y = V(t(X) = s(X))$ over L_l . Then*

1. a point P belongs to all Y_σ iff $P = (a, a, \dots, a)$ for some $a \in L_l$;

2.

$$Y_\sigma \setminus \bigcup_{\sigma' \neq \sigma} Y_{\sigma'} = \{P_\sigma\}$$

(it follows that the decomposition (4) is redundant, i.e. each point of $Y \setminus \bigcup_{\sigma} \{P_\sigma\}$ is covered by at least two irreducible components);

3. all irreducible components are isomorphic to each other;

4. $|Y_\sigma| = \binom{2^l-1}{l}$ for each σ .

Proof. 1. Obviously, $P = (a, a, \dots, a)$ satisfies all systems \mathbf{S}_σ , so $P \in \bigcap_{\sigma} Y_\sigma$.

Let us consider a point $Q = (q_1, q_2, \dots, q_n)$ with $q_i < q_j$. It is clear that Q does not satisfy any set Y_σ with $x_i \geq x_j$. Thus, $Q \notin \bigcap_{\sigma} Y_\sigma$.

2. In Lemma 4.3 we proved $P_\sigma \in Y_\sigma$. By the definition, only the point P_σ makes all inequalities \leq of the system \mathbf{S}_σ strict. Thus, for any point $P = (p_1, p_2, \dots, p_n) \in Y_\sigma \setminus \{P_\sigma\}$ there exists an equation $x_i \leq x_j \in \mathbf{S}_\sigma$ such that $p_i = p_j$. Below we find an irreducible partition σ' with $P \in Y_{\sigma'}$.

Let $\sigma = (X_1, X_2, \dots, X_l)$, $x_i \in X_{i'}$ and without loss of generality one can assume that $x_j \in X_{i'+1}$. If $i' \neq 1$ we put $\sigma' = (X'_1, X'_2, \dots, X'_l)$ where

$$X'_k = \begin{cases} X_k & \text{if } k \neq i', k \neq i' + 1, \\ (X_{i'+1} \setminus \{x_j\}) \cup \{x_i\} & \text{if } k = i' + 1, \\ (X_{i'} \setminus \{x_i\}) \cup \{x_j\} & \text{if } k = i' \end{cases} \quad (5)$$

Since $X'_1 = X_1$, σ' is a Y -irreducible partition. The system $\mathbf{S}_{\sigma'}$ contains $x_j \leq x_i$ instead of $x_i \leq x_j \in \mathbf{S}_\sigma$. Since other relations in the systems $\mathbf{S}_{\sigma'}, \mathbf{S}_\sigma$ are the same, $P \in V(\mathbf{S}_{\sigma'}) = Y_{\sigma'}$.

Suppose now $i' = 1$. Without loss of generality we assume $x_i \in \text{Var}(t)$. By the definition of a Y -irreducible partition, there exists a variable $x_k \in X_1 \cap \text{Var}(s)$. If $x_j \in \text{Var}(t)$ we can define σ' by (5). In this case X'_1 contains variables $x_j \in \text{Var}(t)$, $x_k \in \text{Var}(s)$, so σ' is an Y -irreducible partition and $P \in Y_{\sigma'}$. Otherwise ($x_j \in \text{Var}(s)$), one can take x_k instead x_i and repeat all reasonings above.

3. The statement immediately follows from Lemma 4.1.

4. For $\sigma = (X_1, X_2, \dots, X_l)$ the number $|Y_\sigma|$ is equal to the number of sequences $X_1 \leq X_2 \leq \dots \leq X_l$ with $X_i \in \{a_1, a_2, \dots, a_l\}$. According to combinatorics, the number of such monotone sequences is $\binom{2^l-1}{l}$. □

5 Average number of irreducible components

Let $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ be the Stirling number of the second kind. By the definition, $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ is the number of all partitions of an n -element set into m non-empty unlabelled subsets. The number $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}^* = m! \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ obviously equals the number of all partitions of n -element set into m labelled non-empty subsets. Thus, there are exactly $\left\{ \begin{smallmatrix} n \\ l \end{smallmatrix} \right\}^*$ ordered partitions $\sigma = (X_1, X_2, \dots, X_l)$ of the set of variables X , $|X| = n$ into l equivalence classes. An ordered partition $\sigma = (X_1, X_2, \dots, X_l)$ is not Y -irreducible if either

$X_1 \subseteq \text{Var}(t) \setminus \text{Var}(s)$ or $X_1 \subseteq \text{Var}(s) \setminus \text{Var}(t)$ For a (k_1, k_2) -equation $t(X) = s(X)$ there exists

$$\sum_{i=1}^{k_1} \binom{k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^*$$

partitions σ with $X_1 \subseteq \text{Var}(t) \setminus \text{Var}(s)$. Similarly, there exist

$$\sum_{i=1}^{k_2} \binom{k_2}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^*$$

partitions σ with $X_1 \subseteq \text{Var}(s) \setminus \text{Var}(t)$.

By Theorem 4.4, for a (k_1, k_2) -equation $t(X) = s(X)$ the number of irreducible components (Y -irreducible partitions) equals

$$\text{Irr}(k_1, k_2, n, l) = \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{i=1}^{k_1} \binom{k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* - \sum_{i=1}^{k_2} \binom{k_2}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^*. \quad (6)$$

The average number of irreducible components of algebraic sets defined by equations from $Eq(n)$ is

$$\begin{aligned} \overline{\text{Irr}}(n, l) &= \frac{\sum_{(k_1, k_2) \in K_n} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n, l)}{\#Eq(n)} = \\ &= \frac{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n, l) - \#Eq(0, n, n) \text{Irr}(0, n, n, l)}{\#Eq(n)} \end{aligned}$$

Below we compute $\overline{\text{Irr}}$ using the following denotations:

1. $A \stackrel{(1)}{=} B$: an expression B is obtained from A by the binomial theorem

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

2. $A \stackrel{(2)}{=} B$: an expression B is obtained from A by the following identity of binomial coefficients

$$\binom{a}{b} \binom{b}{c} = \binom{a}{c} \binom{a-c}{b-c}.$$

3. $A \stackrel{(3)}{=} B$: an expression B is obtained from A by the recurrence relation of Stirling numbers

$$\left\{ \begin{matrix} a+1 \\ b \end{matrix} \right\} = b \left\{ \begin{matrix} a \\ b \end{matrix} \right\} + \left\{ \begin{matrix} a \\ b-1 \end{matrix} \right\}.$$

4. $A \stackrel{(4)}{=} B$: an expression B is obtained from A by the following identity of Stirling numbers

$$\left\{ \begin{matrix} a+1 \\ b+1 \end{matrix} \right\} = \sum_{i=0}^a \binom{a}{i} \left\{ \begin{matrix} i \\ b \end{matrix} \right\}.$$

Remark that in the last formula one can change the sum $\sum_{i=0}^a$ to $\sum_{i=c}^a$ ($c < b$), since $\left\{ \begin{matrix} c \\ b \end{matrix} \right\} = 0$ for $c < b$.

We have

$$\begin{aligned} \#Eq(0, n, n) \text{Irr}(0, n, n, l) &= \binom{n}{0} \binom{n}{n} \left(\left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{i=1}^n \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \right) = \\ \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{i=1}^n \binom{n}{n-i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* &= \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{j=0}^{n-1} \binom{n}{j} \left\{ \begin{matrix} j \\ l-1 \end{matrix} \right\}^* = \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - (l-1)! \sum_{j=0}^{n-1} \binom{n}{j} \left\{ \begin{matrix} j \\ l-1 \end{matrix} \right\} \stackrel{(4)}{=} \\ \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - (l-1)! \left(\left\{ \begin{matrix} n+1 \\ l \end{matrix} \right\} - \left\{ \begin{matrix} n \\ l-1 \end{matrix} \right\} \right) &\stackrel{(3)}{=} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - (l-1)! \left\{ \begin{matrix} n \\ l \end{matrix} \right\} = 0, \end{aligned}$$

$$\begin{aligned} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n) &= \\ \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \left(\left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{i=1}^{k_1} \binom{k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* - \sum_{i=1}^{k_2} \binom{k_2}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \right) &= \\ \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \sum_{i=1}^{k_1} \binom{k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* - \\ \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \sum_{i=1}^{k_2} \binom{k_2}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* &= S_1 - S_2 - S_3, \end{aligned}$$

where

$$S_1 = \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* \sum_{k_1=0}^{n-1} \binom{n}{k_1} 2^{n-k_1} \stackrel{(1)}{=} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* (3^n - 1),$$

$$\begin{aligned} S_2 &\stackrel{(2)}{=} \sum_{k_1=0}^{n-1} \sum_{i=1}^{k_1} \binom{n}{k_1} \binom{k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} \stackrel{(1)}{=} \\ \sum_{k_1=0}^{n-1} \sum_{i=1}^{k_1} \binom{n}{i} \binom{n-i}{k_1-i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 2^{n-k_1} &= \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \sum_{k_1=i}^{n-1} \binom{n-i}{k_1-i} 2^{n-k_1} = \\ \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \sum_{j=0}^{n-i-1} \binom{n-i}{j} 2^{n-i-j} &= \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \left(\sum_{j=0}^{n-i} \binom{n-i}{n-i-j} 2^{n-i-j} - 1 \right) \stackrel{(1)}{=} \\ \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* (3^{n-i} - 1). \end{aligned}$$

Computing

$$\begin{aligned} \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* &= (l-1)! \sum_{j=1}^{n-1} \binom{n}{j} \left\{ \begin{matrix} j \\ l-1 \end{matrix} \right\} \stackrel{(4)}{=} (l-1)! \left(\left\{ \begin{matrix} n+1 \\ l \end{matrix} \right\} - \left\{ \begin{matrix} n \\ l-1 \end{matrix} \right\} \right) \stackrel{(3)}{=} \\ (l-1)! \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* &= \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*, \end{aligned}$$

we obtain

$$S_2 = \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 3^{n-i} - \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* = S(n, l) - \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*,$$

where

$$S(n, l) = \sum_{i=1}^{n-1} \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 3^{n-i}.$$

Let us compute

$$\begin{aligned} S_3 &= \sum_{k_1=0}^{n-1} \sum_{i=1}^{n-k_1} \sum_{k_2=i}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{i} \binom{n-k_1-i}{k_2-i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* = \\ &= \sum_{k_1=0}^{n-1} \sum_{i=1}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* \sum_{k_2=i}^{n-k_1} \binom{n-k_1-i}{k_2-i} \stackrel{(1)}{=} \sum_{k_1=0}^{n-1} \sum_{i=1}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 2^{n-k_1-i} \stackrel{(2)}{=} \\ &= \sum_{k_1=0}^{n-1} \sum_{i=1}^{n-k_1} \binom{n}{i} \binom{n-i}{n-k_1-i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 2^{n-k_1-i} = \sum_{i=1}^n \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 2^{n-i} \sum_{k_1=0}^{n-i} \binom{n-i}{k_1} 2^{-k_1} \stackrel{(1)}{=} \\ &= \sum_{i=1}^n \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 2^{n-i} \left(1 + \frac{1}{2} \right)^{n-i} = \sum_{i=1}^n \binom{n}{i} \left\{ \begin{matrix} n-i \\ l-1 \end{matrix} \right\}^* 3^{n-i} = S(n, l) + \binom{n}{n} \left\{ \begin{matrix} n-n \\ l-1 \end{matrix} \right\}^* = S(n, l). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \overline{\text{Irr}}(n, l) &= \frac{S_1 - S_2 - S_3 - 0}{3^n - 2} = \frac{\left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* (3^n - 1) - (S(n, l) - \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*) - S(n, l)}{3^n - 2} = \\ &= \frac{3^n \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - 2S(n, l)}{3^n - 2}. \quad (7) \end{aligned}$$

Let us compute $\overline{\text{Irr}}(n, 2)$ using the following identities of the Stirling numbers

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1, \quad \left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1.$$

We have

$$S(n, 2) = \sum_{i=1}^{n-1} \binom{n}{i} \cdot 1 \cdot 3^{n-i} = \sum_{i=1}^{n-1} \binom{n}{i} 3^{n-i} \stackrel{(1)}{=} 4^n - 3^n - 1,$$

therefore

$$\overline{\text{Irr}}(n, 2) = \frac{3^n \cdot 2(2^{n-1} - 1) - 2(4^n - 3^n - 1)}{3^n - 2} = \frac{6^n - 2 \cdot 4^n + 2}{3^n - 2}. \quad (8)$$

In particular, $n = 3$ gives

$$\overline{\text{Irr}}(3, 2) = \frac{6^3 - 2 \cdot 4^3 + 2}{3^3 - 2} = \frac{90}{25} = 3.6 \quad (9)$$

that coincides with (3).

The following statement gives the estimation of $\overline{\text{Irr}}(n, l)$.

Proposition 5.1. *The number $\overline{\text{Irr}}(n, l)$ satisfies*

$$\frac{1}{3} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* \leq \overline{\text{Irr}}(n, l) \leq \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*$$

Proof. One can bound $S(n, l)$ as follows

$$S(n, l) \leq 3^{n-1} \sum_{i=1}^{n-1} \binom{n}{j} \left\{ \begin{matrix} j \\ l-1 \end{matrix} \right\}^* \stackrel{(4)}{=} 3^{n-1} (l-1)! \left(\left\{ \begin{matrix} n+1 \\ l \end{matrix} \right\} - \left\{ \begin{matrix} n \\ l-1 \end{matrix} \right\} \right) \stackrel{(3)}{=} 3^{n-1} (l-1)! l \left\{ \begin{matrix} n \\ l \end{matrix} \right\} = 3^{n-1} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*,$$

and similarly

$$S(n, l) \geq 3 \sum_{i=1}^{n-1} \binom{n}{j} \left\{ \begin{matrix} j \\ l-1 \end{matrix} \right\}^* = 3 \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*.$$

Thus,

$$\overline{\text{Irr}}(n, l) \leq \frac{3^n \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - 2 \cdot 3 \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*}{3^n - 2} = \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* \frac{3^n - 6}{3^n - 2} \leq \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*,$$

and

$$\overline{\text{Irr}}(n, l) \geq \frac{3^n \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* - 2 \cdot 3^{n-1} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*}{3^n - 2} = \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* \frac{3^n - 2 \cdot 3^{n-1}}{3^n - 2} \geq \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* \frac{3^n - 2 \cdot 3^{n-1}}{3^n} = \frac{1}{3} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^*.$$

□

Proposition 5.2. *For a fixed l and $n \rightarrow \infty$ we have the asymptotic equivalence*

$$\overline{\text{Irr}}(n, l) \sim l^n.$$

Proof. Using the following explicit formula for Stirling numbers

$$\left\{ \begin{matrix} n \\ l \end{matrix} \right\} = \frac{1}{l!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} j^n,$$

we obtain $\left\{ \begin{matrix} n \\ l \end{matrix} \right\} \sim l^n$ for fixed l and $n \rightarrow \infty$. By Proposition 5.1, we have

$$\overline{\text{Irr}}(n, l) \sim \left\{ \begin{matrix} n \\ l \end{matrix} \right\}^* = l! \left\{ \begin{matrix} n \\ l \end{matrix} \right\} \sim l! l^n \sim l^n.$$

□

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